Quantum corrections for translation-invariant renormalizable non-commutative ϕ^4 theory

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ABSTRACT: In this paper we elaborate on the translation-invariant renormalizable ϕ^4 theory in 4-dimensional non-commutative space which was recently introduced by the Orsay group. By explicitly performing Feynman graph calculations at one loop and higher orders we illustrate the mechanism which overcomes the UV/IR mixing problem and ultimately leads to a renormalizable model. The obtained results show that the IR divergences are also suppressed in the massless case, which is of importance for the gauge field theoretic generalization of the scalar field model.

KEYWORDS: Non-Commutative Geometry, Renormalization Regularization and Renormalons.

Contents

1. Introduction

The simplest generalization of the ϕ^4 theory from ordinary space to non-commutative space is given by the action [1]–[3]

$$S_{\text{naive}}[\phi] \equiv \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2} \left(\partial^\mu \phi \star \partial_\mu \phi + m^2 \phi \star \phi \right) + \frac{\lambda}{4!} \, \phi \star \phi \star \phi \star \phi \right] . \tag{1.1}$$

Here, ϕ denotes a relativistic scalar field in 4-dimensional Euclidean space, and the non-commutativity is implemented by the Weyl-Moyal star product [4],

$$[x^\mu \stackrel{\star}{,} x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu} , \tag{1.2}$$

where the parameters $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are real constants. In the following, we assume that the deformation matrix $(\theta_{\mu\nu})$ has the simple block-diagonal form

$$(\theta_{\mu\nu}) = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \qquad \text{with } \theta \in \mathbb{R} . \tag{1.3}$$

We note that the parameter θ is necessarily quite small on physical grounds, e.g. see reference [5] for a recent discussion of experimental aspects.

By now it is well established that the non-commutative model described by the action (1.1) is not renormalizable due to the fact that it suffers from the infamous UV/IR mixing problem that plagues Moyal deformed field theories [1, 3]. For this reason, we refer to the action (1.1) as the *naïve model* [6]. A procedure to obtain a renormalizable theory consists of adding a properly chosen term to the action (1.1) in order to overcome the UV/IR mixing problem. In doing so, one clearly changes the initial model, but one obtains a consistent quantum field theory. Recently, three such proposals have been made and each of them has been proved to provide a renormalizable theory (see reference [7] for a short review).

The first proposal, put forward by Grosse and Wulkenhaar [8], consists of adding a harmonic oscillator-like potential of the form $\tilde{x}^2 \phi^2$ (where $\tilde{x}_\mu \equiv \theta_{\mu\nu} x^\nu$). A notable shortcoming of this model is that it breaks translation invariance. We also remark that the corresponding quantum corrections are hard to evaluate due to the occurrence of the Mehler kernel which is quite involved. The generalization of this model to non-commutative gauge theories is not obvious, though some work in this direction has been done [9].

Another proposal, due to Grosse and Vignes-Tourneret [10], consists of adding a non-local term of the form $\frac{\mu}{\theta^4} \left(\int d^4x \, \phi(x) \right)^2$ to the action (1.1). This approach yields a minimalist translation-invariant ϕ^4-theory in non-commutative space.

The third proposal, made by Gurau, Magnen, Rivasseau and Tanasa [6], consists of adding a non-local counterterm $\phi \frac{1}{\theta^2 \Box} \phi$ for the quadratic IR divergence of the naïve model. This procedure provides a solution for the UV/IR mixing problem while maintaining translation invariance.

In the present work, we focus on the latter model, i.e. on the action

$$S[\phi] \equiv \int\limits_{\mathbb{R}^4} d^4x \left[\frac{1}{2} \left(\partial^\mu \phi \star \partial_\mu \phi + m^2 \phi \star \phi - \phi \star \frac{a^2}{\Box} \phi \right) + \frac{\lambda}{4!} \, \phi \star \phi \star \phi \star \phi \right], \qquad (1.4)$$

in Euclidean space. In expression (1.4), the parameter a is assumed to have the form $a = a'/\theta$ where a' represents a real dimensionless constant. One of our motivations is that the non-local term has been generalized [11] to U(1) gauge theories whose renormalizability still remains an open problem. Therefore, it should be quite useful to gain deeper insight into the quantum corrections for the model (1.4) since the multiscale analysis [12] which was used to establish its renormalizability cannot be applied to gauge field theories as it breaks gauge invariance. Accordingly we will study perturbative corrections to the propagators and vertices for the theory (1.4), and in particular the vanishing mass limit which is of interest for gauge theories. In doing so, we will explicitly exhibit the improvements for the quantum theory brought about by the non-local term.

Our paper is organized as follows: After introducing the Feynman rules for the model in the next section, we deal with the renormalization procedure at the one-loop level in sections 3 and 4. Section 5 is devoted to two and more loops, and exhibits the mechanism that removes the potential IR divergences, which ultimately leads to the renormalizability

of the model [6]. In this respect, the scaling behaviour of the propagator determined by the action (1.4) plays an essential role. The appendices describe the techniques for handling the loop calculations and should apply to generic higher loop graphs.

Although our investigations concern the massive theory, we also consider the limit $m \to 0$ in several instances and find well defined results. This issue is of importance for the gauge field theoretic generalization of the model (1.4) which was recently proposed [11] and which is to be further discussed elsewhere [13].

2. Propagator and vertex

Concerning the action (1.4) we recall [11] that the operator $1/\Box$ denotes the Green function associated to the 4-dimensional Laplacian $\Box \equiv \partial^\mu \partial_\mu = \partial_1^2 + \ldots + \partial_4^2$ and that it is given by $-1/k^2$ in momentum space. Accordingly, the propagator in momentum space reads

$$\overset{k}{\longrightarrow} \;\; = \; G(k) = \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} \,. \tag{2.1}$$

Note that this propagator has a "damping" behaviour for vanishing momentum [6],

$$\lim_{k \to 0} G(k) = 0 \,,$$

which allows to avoid potential IR divergences in higher loop graphs (see section 5). This property is due to the non-local term $\phi \frac{a^2}{\Box} \phi$ in the action (1.4) and represents a crucial difference (and improvement) compared to the naïve model (1.1).

In terms of the notation $\tilde{k}_\mu \equiv \theta_{\mu\nu} k^\nu$, the vertex in momentum space has the following form (see reference [2]):

$$\begin{aligned}
&= V(k_1, k_2, k_3, k_4) \\
&= \frac{\lambda}{3} (2\pi)^4 \delta^4 (k_1 + k_2 + k_3 + k_4) \left[\cos\left(\frac{1}{2} k_1 \tilde{k}_2 \right) \cos\left(\frac{1}{2} k_3 \tilde{k}_4 \right) \right. \\
&\quad \left. + \cos\left(\frac{1}{2} k_1 \tilde{k}_3 \right) \cos\left(\frac{1}{2} k_2 \tilde{k}_4 \right) + \cos\left(\frac{1}{2} k_1 \tilde{k}_4 \right) \cos\left(\frac{1}{2} k_2 \tilde{k}_3 \right) \right]. \tag{2.2}
\end{aligned}$$

3. One-loop corrections for propagator and vertex

To start with, we determine the relevant corrections for the 1PI two-point and four-point functions at one-loop level.

3.1 Propagator

The one-loop correction to the propagator, which is described by the Feynman diagrams in figure 1, corresponds to the following integral (including a symmetry factor $1/2$):

$$\Pi(p) = -\frac{\lambda}{6} \int\limits_{\mathbb{R}^4} \frac{d^4 k}{(2\pi)^4} \frac{2 + \cos(k\tilde{p})}{k^2 + m^2 + \frac{a^2}{k^2}} \equiv \Pi^{\text{plan}} + \Pi^{\text{n-pl}}(p) \,. \tag{3.1}$$
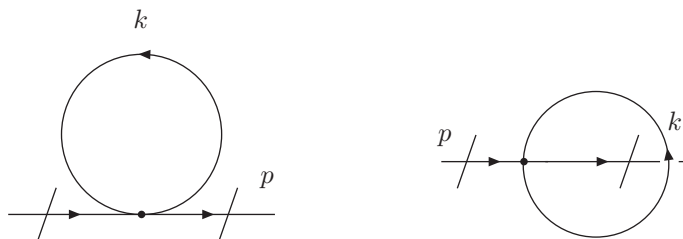
Figure 1: Planar and non-planar one-loop corrections for the propagator

Here, Π^{plan} and $\Pi^{\text{n-pl}}$ denote the *planar* and *non-planar parts,* respectively. We note that

$$\cos(k\tilde{p}) = \frac{1}{2} \sum_{\eta=\pm 1} e^{i\eta k\tilde{p}}, \qquad (3.2)$$

and

$$\frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} = \frac{k^2}{\left(k^2 + \frac{m^2}{2}\right)^2 - M^4} = \frac{1}{2} \sum_{\zeta=\pm 1} \frac{1 + \zeta\frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \zeta M^2}, \qquad (3.3)$$

where $M^2 \equiv \sqrt{\frac{m^4}{4} - a^2}$ (which may be real or purely imaginary depending on the value of a). Taking into account these identities, the *non-planar part* can be evaluated straightforwardly by using Schwinger's exponential parametrization (see appendix A for details):

$$\Pi^{\text{n-pl}}(p) \equiv -\frac{\lambda}{24} \int \frac{d^4k}{(2\pi)^4} \sum_{\eta,\zeta=\pm 1} \frac{1 + \zeta\frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \zeta M^2} e^{i\eta k\tilde{p}}$$

$$= -\frac{\lambda}{48\pi^2} \sum_{\zeta=\pm 1} \left(1 + \zeta\frac{m^2}{2M^2}\right) \sqrt{\frac{\frac{m^2}{2} + \zeta M^2}{\tilde{p}^2}} K_1\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2} + \zeta M^2\right)}\right), \qquad (3.4)$$

where K_1 is the modified Bessel function. The result is finite for $\tilde{p}^2 \neq 0$, i.e. if $\theta \neq 0$ and $p \neq 0$.

In the following, we will focus on the IR behaviour of the model, i.e. the limit $\tilde{p}^2 \to 0$. For small z, the function $\frac{1}{z}K_1(z)$ admits the expansion

$$\frac{1}{z}K_1(z) = \frac{1}{z^2} + \frac{1}{2}\ln z + \frac{1}{2}\left(\gamma_E - \ln 2 - \frac{1}{2}\right) + \frac{z^2}{16}\left(\ln z + \gamma_E - \ln 2 - \frac{5}{4}\right) + \mathcal{O}(z^4), \quad (3.5)$$

where γ_E denotes the Euler-Mascheroni constant. Thus, for $\tilde{p}^2 \ll 1$, the expression (3.4) behaves like

$$\Pi^{\text{n-pl}}(p) = \frac{-\lambda}{6(4\pi)^2}\left[\frac{4}{\tilde{p}^2} + m^2\ln\left(\tilde{p}^2\sqrt{\frac{m^4}{4} - M^4}\right) + \left(M^2 + \frac{m^4}{4M^2}\right)\ln\sqrt{\frac{\frac{m^2}{2} + M^2}{\frac{m^2}{2} - M^2}}\right] + \mathcal{O}(1),$$
$$(3.6)$$

and thereby involves a quadratic IR divergence (and a subleading logarithmic IR divergence). For $a \to 0$ (i.e. $M^2 \to \frac{m^2}{2}$) this result reduces to the one which was previously found [1, 2] for $a = 0$, i.e. for the naïve model.

The integral defining the *planar part* does not contain a phase factor $\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}$ and is therefore UV divergent. It can be regularized by introducing a cutoff Λ and subsequently taking the limit $\tilde{p}^2 \to 0$, as explained in appendix A.1. The final result can be expanded for large values of Λ, yielding

$$
\left(\Pi^{\mathrm{plan}}\right)_{\mathrm{regul.}}(\Lambda) = \frac{-\lambda}{3(4\pi)^2}\left[4\Lambda^2 + m^2\ln\left(\frac{1}{\Lambda^2}\sqrt{\frac{m^4}{4} - M^4}\right)\right.
$$
$$
\left. + \left(M^2 + \frac{m^4}{4M^2}\right)\ln\sqrt{\frac{\frac{m^2}{2} + M^2}{\frac{m^2}{2} - M^2}}\right] + \mathcal{O}(1). \qquad (3.7)
$$

3.2 Vertex

The basic one-loop correction to the vertex is given by the three connected graphs that can be constructed with four external legs [14, 2, 15]:

$$
V_{\text{1-loop}}(p_1, p_2, p_3, p_4) = \frac{1}{3}\left[\frac{1}{2}\,\raisebox{-1em}{}\, + \,\raisebox{-1em}{}\, + \,\raisebox{-1em}{}\,\right]. \qquad (3.8)
$$

This expression can be evaluated by proceeding along the lines of reference [2]: by applying the Feynman rules (2.1) and (2.2), and by taking advantage of the identity (3.3) we find that (3.8) reads

$$
\frac{\lambda^2}{27}\sum_{\zeta,\chi=\pm 1}\int\frac{d^4k}{(2\pi)^4}\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{k^2 + \frac{m^2}{2} + \zeta M^2}\left[\left(1 + \frac{1}{4}\sum_{i=2}^4 \mathrm{e}^{\mathrm{i}k(\tilde{p}_1+\tilde{p}_i)} + \frac{1}{2}\sum_{i=1}^4 \mathrm{e}^{\mathrm{i}k\tilde{p}_i}\right)\times\right.
$$
$$
\times\left(\frac{1}{(p_1+p_2-k)^2 + \frac{m^2}{2} + \chi M^2} + \frac{1}{(p_1+p_3-k)^2 + \frac{m^2}{2} + \chi M^2}\right.
$$
$$
\left.+ \frac{1}{(p_1+p_4-k)^2 + \frac{m^2}{2} + \chi M^2}\right)
$$
$$
+ \frac{3}{4}\left(\frac{\mathrm{e}^{\mathrm{i}k(\tilde{p}_1+\tilde{p}_2)}}{(p_1+p_2-k)^2 + \frac{m^2}{2} + \chi M^2} + \frac{\mathrm{e}^{\mathrm{i}k(\tilde{p}_1+\tilde{p}_3)}}{(p_1+p_3-k)^2 + \frac{m^2}{2} + \chi M^2}\right.
$$
$$
\left.\left.+ \frac{\mathrm{e}^{\mathrm{i}k(\tilde{p}_1+\tilde{p}_4)}}{(p_1+p_4-k)^2 + \frac{m^2}{2} + \chi M^2}\right)\right]. \qquad (3.9)
$$

Thus, we again have an expression involving planar and non-planar parts (the latter involving a phase factor of the form $\mathrm{e}^{\mathrm{i}k\tilde{q}}$). The generic integral for the *non-planar part* is

given by

$$
\begin{aligned}
I(p,q) &\equiv \sum_{\zeta,\chi=\pm 1} \int \frac{d^4 k}{(2\pi)^4} \frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right) e^{ik(\tilde{p}+\tilde{q})}}{\left(k^2+\frac{m^2}{2}+\zeta M^2\right)\left[(p-k)^2+\frac{m^2}{2}+\chi M^2\right]} \\
&= \sum_{\zeta,\chi}\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)\int_0^1 d\xi \frac{e^{i(1-\xi)p\tilde{q}}}{8\pi^2} \\
&\quad \times K_0\left(\sqrt{(\tilde{p}+\tilde{q})^2\left[\xi(1-\xi)p^2+\frac{m^2}{2}+(\chi+\xi(\zeta-\chi))M^2\right]}\right). \quad (3.10)
\end{aligned}
$$

Here, p denotes the total incoming momentum, and q represents one of the variables p_i (see appendix A for calculational details). For small arguments the modified Bessel function K_0 can be expanded according to

$$
K_0(z) = -\ln z + \ln 2 - \gamma_E + \mathcal{O}\left(z^2\right), \quad (3.11)
$$

from which we can derive the following estimation for small external momenta p and q:

$$
\begin{aligned}
|I(p,q)| &\leq \frac{1}{(2\pi)^2}\left| \ln\left(\frac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\frac{m^4}{4}-M^4}\right) + 2\gamma_E - \frac{1}{2}\left(1-\frac{m^4}{4M^4}\right) \right. \\
&\quad \left. + \left(1+\frac{m^4}{4M^4}\right)\frac{m^2}{4M^2}\ln\sqrt{\frac{\frac{m^2}{2}+M^2}{\frac{m^2}{2}-M^2}}\right|. \quad (3.12)
\end{aligned}
$$

The *planar part* of expression (3.8) can again be evaluated by introducing a cut-off Λ (as was discussed for the propagator in appendix A.1): the final result directly follows from (3.10) and (3.12) by replacing $(\tilde{p}+\tilde{q})^2$ with $1/\Lambda^2$.

4. One-loop renormalization

According to the standard renormalization procedure, the *dressed propagator at one-loop level* is given by

$$
\underset{p}{-\bigcirc-} \equiv \Delta'(p) = \frac{1}{A} + \frac{1}{A}\Sigma(\Lambda,p)\frac{1}{A}, \quad (4.1)
$$

where

$$
A \equiv p^2+m^2+\frac{a^2}{p^2},
$$

$$
\Sigma(\Lambda,p) \equiv \left(\Pi^{\text{plan}}\right)_{\text{regul.}}(\Lambda) + \Pi^{\text{n-pl}}(p).
$$

Since $A \neq 0$, we can apply the formula

$$
\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \mathcal{O}(B^2),
$$

which allows us to rewrite expression (4.1) to order Σ (i.e. to order λ) as

$$\Delta'(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{p^2} - \Sigma(\Lambda, p)} \, . \tag{4.2}$$

The contribution $\Pi^{\text{n-pl}}(p)$ to $\Sigma(\Lambda, p)$ is finite except for vanishing external momentum p. The expansion for small values of $\tilde{p}^2$, as given in Eq. (3.6), reveals a quadratic and a logarithmic IR divergence[1] at $\tilde{p}^2 = 0$. The quadratically divergent term obviously has the same structure as the term $\frac{a^2}{p^2}$ appearing in the bare propagator (2.1). Henceforth, we will absorb it by a finite renormalization of the parameter a^2.

From the expansions (3.6) and (3.7) it follows that, to order λ, we have

$$\Delta'(p) = \frac{Z}{p^2 + m_r^2 + \frac{a_r^2}{p^2} + f(p^2)} \, , \tag{4.3}$$

where

$$
\begin{aligned}
Z &\equiv 1 + \lambda\alpha\theta^2 \, , \qquad\qquad (\alpha \in \mathbb{R}) \, , \\
m_r^2 &\equiv m^2 + \frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2 + m^2 \ln\left(\frac{1}{\Lambda^2}\sqrt{\frac{m^4}{4} - M^4} \right) \right] + \text{regular for } \Lambda \to \infty \, , \\
a_r^2 &\equiv a^2 + \lambda\left[\frac{2}{3(4\pi\theta)^2} + \alpha a^2\theta^2 \right] \, , \\
f(p^2) &\equiv \frac{\lambda}{6(4\pi)^2}\left[m^2 \ln\left(\theta^2 p^2 \right) + \mathcal{O}((\theta p)^4) \right] \, .
\end{aligned}
\tag{4.4}
$$

The quantities m_r and a_r represent the renormalized mass and a-parameter to one-loop order, and the function $f(p^2)$ is analytic for $\theta \neq 0$ and $p^2 > 0$. The expression Z amounts to a finite wave function renormalization[2]. The logarithmic singularity of $f(p^2)$ for vanishing external momentum p represents a mild divergence which is unproblematic for the amplitudes (see also the multiscale analysis [6]). The constant α appearing in Z and in

$$a_r'^2 \equiv \theta^2 a_r^2 = a'^2 + \lambda\left[\frac{2}{3(4\pi)^2} + \alpha\theta^2 a'^2 \right] \tag{4.5}$$

is determined by the numerical factor that occurs in the expansion of $\Sigma(\Lambda, p)$ at order $\tilde{p}^2$ (see equations (3.5) and (3.6)). We have

$$\alpha\theta^2 = \frac{2}{3(16\pi)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E \right)\left(\theta^2 m^4 - a'^2 \right) \, , \tag{4.6}$$

---

[1]In this respect, we should emphasize that these IR divergences are fundamentally different from the ones encountered in quantum field theories on commutative space since they are tied to the UV divergences and only appear in non-planar diagrams which are not present in usual QFT [4]. Thus, these divergences cannot be regularized by introducing an infrared regulator (like an additional mass parameter).

[2]For the ϕ^4-theory on commutative space, there is no wave function renormalization at one-loop order, but this is a peculiarity of this theory [14].

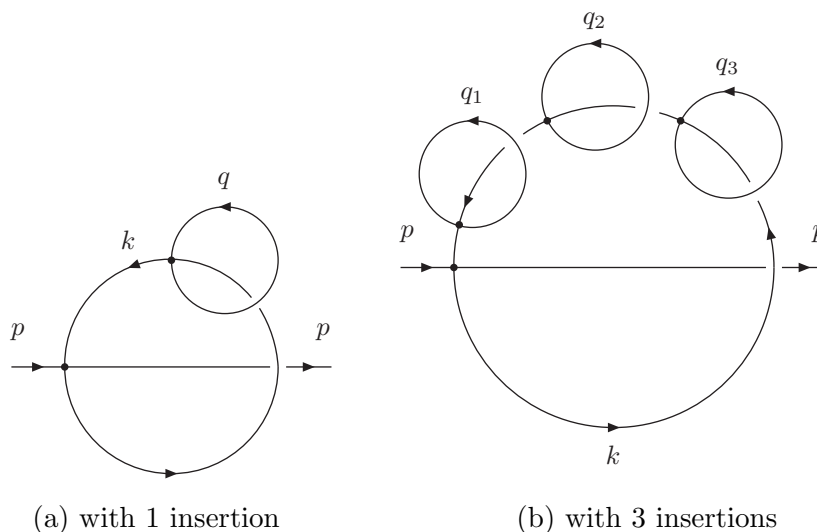(a) with 1 insertion   (b) with 3 insertions

Figure 2: Non-planar 2-loop and 4-loop graphs.

which is positive for $\theta^2 m^4 > a'^2$. However, even in the case where $\alpha < 0$, the one-loop renormalized parameter $a_r'^2$ is positive provided

$$a'^2 < \frac{1}{2}\left(\frac{B}{\lambda A} + \theta^2 m^4\right) + \sqrt{\frac{1}{4}\left(\frac{B}{\lambda A} + \theta^2 m^4\right)^2 + B}\,,$$

$$\text{where} \qquad A \equiv \frac{2}{3(4\pi)^2}\,, \quad B \equiv \frac{16}{\ln 2 + \frac{5}{4} - \gamma_E} > 0\,. \tag{4.7}$$

Since θ is necessarily quite small on physical grounds, the dominating factor in the previous inequality is $1/\lambda$. Hence, even for $m = 0$, the parameter $a_r'^2$ is positive for small values of the coupling constant λ (more precisely for $a'^2 < 10^3/\lambda$).

The renormalized coupling constant λ_r at one-loop order is obtained by considering the planar part of (3.8). One finds an expression which is similar to the one in the commutative theory. The non-planar part of (3.8) again involves a logarithmic singularity (see equation (3.12)).

5. Two and higher loops

In order to exhibit how the non-local term $\phi\frac{1}{\theta^2\Box}\phi$ improves the IR behaviour of the naïve model at the higher loop level, we consider a non-planar tadpole graph with non-planar insertions (see figure 2a for one insertion and figure 2b for several insertions).

Since we are only concerned about the IR divergences, we limit ourselves to the first (i.e. most singular) term in the expansion (3.6) of $\Pi^{\text{n-pl}}$, hence [1] we consider the approximation $\Pi^{\text{n-pl}}(k) \propto 1/\tilde{k}^2$. Within this approximation, a graph with n non-planar insertions is

described by the expression

$$\Pi^{n \text{ npl-ins.}}(p) \equiv \lambda^2 \sum_{\eta=\pm 1} \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{\left(\tilde{k}^2\right)^n \left[k^2 + m^2 + \frac{a^2}{k^2}\right]^{n+1}} \,. \tag{5.1}$$

For the naïve model (where $a = 0$), the integral (5.1) involves an IR divergence for $n \geq 2$, because the integrand behaves like $(k^2)^{-n}$ for $k^2 \to 0$. In contrast, for the model under consideration (where $a \neq 0$), the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left[\frac{a^2}{k^2}\right]^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}} \,. \tag{5.2}$$

Thus, the propagator (2.1) "damps" the IR-dangerous insertions and therefore cures potential IR problems in the integral (5.1). This is a nice demonstration of the mechanism leading to the renormalizability of the present model. In this respect we recall that its renormalizability has been proved quite generally in reference [6] using multiscale analysis.

In the following we will present a more detailed mathematical analysis of the IR behaviour of the graph with n non-planar insertions. In fact, the integral (5.1) can be evaluated by the same techniques as those applied in the previous section (see appendices B and C for details). For $n = 1$, one finds that

$$\Pi^{1 \text{ npl-ins.}}(p) = \frac{-\lambda^2}{16\pi^2\theta^2 M^6} \left\{ m^2 \left[\sqrt{\frac{m_+^2}{\tilde{p}^2}} K_1\left(\sqrt{m_+^2\tilde{p}^2}\right) - \sqrt{\frac{m_-^2}{\tilde{p}^2}} K_1\left(\sqrt{m_-^2\tilde{p}^2}\right) \right] \right.$$
$$\left. + M^2 \left[m_+^2 K_0\left(\sqrt{m_+^2\tilde{p}^2}\right) + m_-^2 K_0\left(\sqrt{m_-^2\tilde{p}^2}\right) \right] \right\}, \tag{5.3}$$

where $m_\pm^2 \equiv \frac{m^2}{2} \pm M^2$. For generic n, we get $\Pi^{n \text{ npl-ins.}}(p) = \lambda^2 \sum_{\eta=\pm 1} J_n(p)$ with $J_n(p)$ given by the integral (C.6). By expanding the expressions for $n = 1$ and $n = 2$ for small external momentum $\tilde{p}^2$, one obtains

$$\Pi^{1 \text{ npl-ins.}}(p) = \frac{\lambda^2}{16\pi^2\theta^2 M^6} \left[\left(M^4 - \frac{m^4}{4} \right) \ln\sqrt{\frac{m_+^2}{m_-^2}} + M^2\frac{m^2}{2} \right] + \mathcal{O}(\tilde{p}^2), \tag{5.4}$$

$$\Pi^{2 \text{ npl-ins.}}(p) = \frac{\lambda^2}{256\pi^2\theta^4} \left[\frac{3m^4 - 4M^4}{M^{10}} \ln\sqrt{\frac{m_+^2}{m_-^2}} - 6\frac{m^2}{M^8} \right] + \mathcal{O}(\tilde{p}^2).$$

We note that the latter result diverges in the limit $a \to 0$. This fact again illustrates how the propagator (2.1) regularizes graphs which diverge in the naïve model.

In the limit $m \to 0$ (i.e. for a massless field), the expressions (5.4) reduce to finite quantities:

$$\Pi^{1 \text{ npl-ins.}}(p)\Big|_{m=0} = \frac{\lambda^2}{32\pi|\theta a'|} + \mathcal{O}(\tilde{p}^2),$$
$$\Pi^{2 \text{ npl-ins.}}(p)\Big|_{m=0} = \frac{\lambda^2}{128\pi|\theta a'^3|} + \mathcal{O}(\tilde{p}^2). \tag{5.5}$$

Henceforth, in contrast to the naïve model (e.g. see reference [2]), the higher-loop graphs in figure 2b do *not* diverge for $m \to 0$. In other words, the IR divergent insertion $1/\tilde{k}^2$ does not cause any harm in these higher-loop graphs, even for a massless field. This is an important feature for the gauge field theoretic generalization of the model (1.4) which was introduced in reference [11], and which is to be addressed elsewhere [13].

6. Concluding remarks

Concerning the quantum corrections for the model under consideration, we note the quite recent work [16–18] which is complementary to our study: the second work is devoted to the calculation of the one-loop beta function and to the parametric representation of Feynman amplitudes (according to the topology of the considered graphs). The approach of references [16, 17] is based on the assumption that $4a^2 < m^4$ whereas our analysis does not require any restrictions on the parameters of the theory (as was mentioned in section 3).

While some of the obtained expressions allow for a smooth limit $a \to 0$ towards the corresponding results of the naïve (non-renormalizable) theory, this is — as expected — not the case for the higher loop corrections.

As is apparent from the calculations outlined in the appendices, the Schwinger parametrization is quite convenient for determining the quantum corrections for the scalar field model. Since the gauge field theoretic generalization of this model extends the propagator for a (massless) scalar field, the same techniques should allow us to tackle the problem of IR divergences and discuss the issue of renormalizability for translation-invariant gauge field theories on non-commutative space.

Acknowledgments

A. 1-loop integrals

A.1 Correction to the propagator

By virtue of equations (3.1)–(3.3), the non-planar part of the one-loop correction to the propagator reads $\Pi^{\text{n-pl}}(p) = -\frac{\lambda}{24} \sum_{\eta=\pm 1} I(p)$ with

$$I(p) \equiv \sum_{\zeta=\pm 1} \left(1 + \zeta \frac{m^2}{2M^2} \right) \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{k^2 + \frac{m^2}{2} + \zeta M^2} \, . \tag{A.1}$$

For $m > 0$ and $a \neq 0$, the combination $\frac{m^2}{2} + \zeta M^2$ has a positive real part so that we can use the Schwinger parametrization:

$$\frac{1}{k^2 + \frac{m^2}{2} + \zeta M^2} = \int\limits_0^\infty d\alpha \, \mathrm{e}^{-\alpha(k^2 + \frac{m^2}{2} + \zeta M^2)} . \tag{A.2}$$

The integral over k can be carried out after completing the square in the exponent:

$$\begin{aligned} I(p) &= \sum_\zeta \left(1 + \zeta \frac{m^2}{2M^2} \right) \int \frac{d^4 k}{(2\pi)^4} \int\limits_0^\infty d\alpha \exp\left[-\alpha \left(k^2 - \frac{\mathrm{i}\eta k\tilde{p}}{\alpha} \right) - \alpha \left(\frac{m^2}{2} + \zeta M^2 \right) \right] \\ &= \sum_\zeta \frac{1 + \zeta \frac{m^2}{2M^2}}{(4\pi)^2} \int\limits_0^\infty \frac{d\alpha}{\alpha^2} \exp\left[-\frac{\tilde{p}^2}{4\alpha} - \alpha \left(\frac{m^2}{2} + \zeta M^2 \right) \right] . \end{aligned} \tag{A.3}$$

Although it is not necessary here, we introduce a cutoff Λ into $I(p)$ by making the replacement $\tilde{p}^2 \to \tilde{p}^2 + \frac{1}{\Lambda^2}$. In fact [2], this will allow us to evaluate the planar part Π^{pl} below. The integral (A.3) can be looked up [19] and the result depends upon the modified Bessel function $K_{-1} = K_1$:

$$I_{\mathrm{regul.}}(p, \Lambda) = \sum_\zeta \frac{1 + \zeta \frac{m^2}{2M^2}}{(2\pi)^2} \sqrt{\frac{\frac{m^2}{2} + \zeta M^2}{\tilde{p}^2 + \frac{1}{\Lambda^2}}} \, K_1\left(\sqrt{\left(\tilde{p}^2 + \frac{1}{\Lambda^2} \right) \left(\frac{m^2}{2} + \zeta M^2 \right)} \right) . \tag{A.4}$$

Taking the limit $\Lambda \to \infty$ leads to the result (3.4) for the non-planar part $\Pi^{\mathrm{n\text{-}pl}}(p)$. The result (3.7) for the regularized planar part $\left(\Pi^{\mathrm{plan}} \right)_{\mathrm{regul.}}(\Lambda)$ also follows from the expression (A.4) by taking the limit $\tilde{p}^2 \to 0$.

A.2 Vertex correction

The non-planar integrals in the vertex correction (3.9) have the following form:

$$\begin{aligned} I(p, q) &\equiv \sum_{\zeta, \chi} \int \frac{d^4 k}{(2\pi)^4} \frac{\left(1 + \zeta \frac{m^2}{2M^2} \right) \left(1 + \chi \frac{m^2}{2M^2} \right) \mathrm{e}^{\mathrm{i}k(\tilde{p}+\tilde{q})}}{\left(k^2 + \frac{m^2}{2} + \zeta M^2 \right) \left((p-k)^2 + \frac{m^2}{2} + \chi M^2 \right)} \\ &= \sum_{\zeta, \chi} \left(1 + \zeta \frac{m^2}{2M^2} \right) \left(1 + \chi \frac{m^2}{2M^2} \right) \int \frac{d^4 k}{(2\pi)^4} \int\limits_0^\infty d\alpha \int\limits_0^\infty d\beta \exp\left[-(\alpha+\beta)\frac{m^2}{2} \right. \\ &\quad \left. -(\alpha+\beta)\left(k^2 - \frac{\mathrm{i}k(\tilde{p}+\tilde{q}) + 2\beta kp}{\alpha+\beta} \right) - \alpha\zeta M^2 - \beta\left(p^2 + \chi M^2 \right) \right]. \end{aligned} \tag{A.5}$$

After carrying out the integration over k, and performing the change of variables $(\alpha, \beta) \to (\lambda, \xi)$ with $\alpha = \lambda\xi$ and $\beta = \lambda(1-\xi)$ (where $\lambda \in [0, \infty[$ and $\xi \in [0,1]$), one obtains an integral over λ,

$$I(p, q) = \frac{1}{(4\pi)^2} \sum_{\zeta, \chi} \left(1 + \zeta \frac{m^2}{2M^2} \right) \left(1 + \chi \frac{m^2}{2M^2} \right) \int\limits_0^1 d\xi \int\limits_0^\infty \frac{d\lambda}{\lambda} \tag{A.6}$$

$$\times \exp\left\{ -\frac{(\tilde{p}+\tilde{q})^2}{4\lambda} + \mathrm{i}(1-\xi)p\tilde{q} - \lambda\left[\xi(1-\xi)p^2 + \frac{m^2}{2} + (\chi + \xi(\zeta - \chi)) M^2 \right] \right\},$$

which can be expressed [19] in terms of the modified Bessel function K_0, see Eq. (3.10).

B. 2-loop integrals

We use the decomposition (3.3) and the Schwinger parametrization to evaluate the integral (5.1) for $n = 1$. Hence $\Pi^{1 \text{ npl-ins.}}(p) = \lambda^2 \sum_{\eta = \pm 1} J_1(p)$ with

$$
\begin{aligned}
J_1(p) &\equiv \frac{1}{4} \sum_{\zeta, \chi} \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\eta k \tilde{p}}}{\tilde{k}^2} \frac{1 + \zeta \frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \zeta M^2} \frac{1 + \chi \frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \chi M^2} \\
&= \frac{1}{4\theta^2} \sum_{\zeta, \chi} \left(1 + \zeta \frac{m^2}{2M^2} \right) \left(1 + \chi \frac{m^2}{2M^2} \right) \int \frac{d^4 k}{(2\pi)^4} \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \\
&\quad \times \exp \left[-(\alpha + \beta + \gamma) k^2 - (\alpha + \beta) \frac{m^2}{2} - (\alpha \zeta + \beta \chi) M^2 + \mathrm{i}\eta k \tilde{p} \right] . \quad \text{(B.1)}
\end{aligned}
$$

After carrying out the integration over k and after the change of variables

$$
\begin{aligned}
(\alpha, \beta, \gamma) &\to (\lambda, \xi, \sigma) \\
\text{with} \quad \alpha &= \lambda \xi \sigma , \\
\beta &= \lambda (1 - \xi) \sigma , \\
\gamma &= \lambda (1 - \sigma) , \\
\text{and} \quad \lambda &\in [0, \infty[, \ \xi \in [0, 1], \ \sigma \in [0, 1] , \quad \text{(B.2)}
\end{aligned}
$$

one obtains

$$
\begin{aligned}
J_1(p) &= \sum_{\zeta, \chi} \frac{\left(1 + \zeta \frac{m^2}{2M^2} \right) \left(1 + \chi \frac{m^2}{2M^2} \right)}{4\theta^2 (4\pi)^2} \int_0^\infty d\lambda \int_0^1 d\xi \int_0^1 d\sigma \, \sigma \\
&\quad \times \exp \left[-\frac{\tilde{p}^2}{4\lambda} - \lambda \sigma \left(\frac{m^2}{2} + \xi \zeta M^2 + (1 - \xi) \chi M^2 \right) \right] \\
&= \frac{1}{\theta^2 (4\pi)^2} \int_0^\infty d\lambda \int_0^1 d\xi \int_0^1 d\sigma \, \sigma \left[\cosh \left(\lambda \sigma \xi M^2 \right) - \frac{m^2}{2M^2} \sinh \left(\lambda \sigma \xi M^2 \right) \right] \\
&\quad \times \left[\cosh \left(\lambda \sigma (1 - \xi) M^2 \right) - \frac{m^2}{2M^2} \sinh \left(\lambda \sigma (1 - \xi) M^2 \right) \right] \mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda} - \lambda \sigma \frac{m^2}{2}} . \quad \text{(B.3)}
\end{aligned}
$$

After integrating out ξ and σ, one is left with sums of integrals over λ which are again given by modified Bessel functions, see Eq. (5.3).

C. n-loop integrals

The calculation proceeds along the lines of the 2-loop integral discussed in appendix B.

The integral (5.1) is given by $\Pi^{n\text{ npl-ins.}}(p) = \lambda^2 \sum\limits_{\eta=\pm 1} J_n(p)$ with

$$J_n(p) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\eta k\tilde{p}}}{\left(\tilde{k}^2\right)^n \left[k^2 + m^2 + \frac{a^2}{k^2}\right]^{n+1}}$$

$$= \frac{1}{2^{n+1}\theta^{2n}} \sum_{\zeta_1,\ldots,\zeta_{n+1}=\pm 1} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\eta k\tilde{p}}}{(k^2)^n} \prod_{i=1}^{n+1} \left(\frac{1 + \zeta_i \frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \zeta_i M^2}\right). \tag{C.1}$$

We need a total of $n+2$ Schwinger parameters α_i to parameterize the denominators of the integrand [20]:

$$\frac{1}{k^2 + \frac{m^2}{2} + \zeta_i M^2} = \int\limits_0^\infty d\alpha_i\, e^{-\alpha_i\left(k^2 + \frac{m^2}{2} + \zeta_i M^2\right)}, \qquad\qquad \text{for } i \in \{1,\ldots,n+1\},$$

$$\frac{1}{(k^2)^n} = \frac{1}{\Gamma(n)} \int\limits_0^\infty d\alpha_{n+2}\, (\alpha_{n+2})^{n-1} e^{-\alpha_{n+2}k^2}, \qquad \text{for } k^2 > 0. \tag{C.2}$$

We perform the change of variables $(\alpha_1,\ldots,\alpha_{n+2}) \to (\xi_1,\ldots,\xi_{n+1},\lambda)$ with

$$\alpha_1 = \lambda \prod_{i=1}^{n+1} \xi_i, \quad \alpha_2 = \lambda(1-\xi_1) \prod_{i=2}^{n+1} \xi_i, \quad \ldots, \quad \alpha_k = \lambda(1-\xi_{k-1}) \prod_{i=k}^{n+1} \xi_i, \\ \ldots, \quad \alpha_{n+2} = \lambda(1-\xi_{n+1}), \tag{C.3}$$

where $\xi_i \in [0,1]$ and $\lambda \in [0,\infty[$. The integration measure transforms as

$$\prod_{i=1}^{n+2} d\alpha_i = \lambda^{n+1} \prod_{l=1}^{n} (\xi_{l+1})^l\, d\lambda \prod_{j=1}^{n+1} d\xi_j, \tag{C.4}$$

and we have $\sum\limits_{i=1}^{n+2} \alpha_i = \lambda$. The integration over k can be carried out by completing the square in the exponent so that we arrive at

$$J_n = \frac{1}{\theta^{2n}2^{n+1}(4\pi)^2\Gamma(n)} \sum_{\zeta_1,\ldots,\zeta_{n+1}} \prod_{i=1}^{n+1} \left(1 + \zeta_i \frac{m^2}{2M^2}\right) \int\limits_0^\infty d\lambda\, \lambda^{2n-2} \prod_{j=1}^{n+1} \int\limits_0^1 d\xi_j \prod_{l=1}^{n} (\xi_{l+1})^l$$

$$\times (1-\xi_{n+1})^{n-1} \exp\left[-\frac{\tilde{p}^2}{4\lambda} - \lambda\xi_{n+1}\frac{m^2}{2} - (\zeta_1\alpha_1 + \ldots + \zeta_{n+1}\alpha_{n+1})M^2\right], \tag{C.5}$$

where the coefficients $\alpha_1,\ldots,\alpha_{n+1}$ in the exponent are functions of the variables $\lambda,\xi_1,\ldots,\xi_{n+1}$ according to Eq. (C.3). The sum over the ζ_i can be expressed in terms of hyperbolic functions:

$$J_n = \frac{1}{\theta^{2n}(4\pi)^2\Gamma(n)} \prod_{j=1}^{n+1} \int\limits_0^1 d\xi_j\, (1-\xi_{n+1})^{n-1} \prod_{l=1}^{n} (\xi_{l+1})^l \int\limits_0^\infty d\lambda\, \lambda^{2n-2} \tag{C.6}$$

$$\times e^{-\frac{\tilde{p}^2}{4\lambda} - \lambda\xi_{n+1}\frac{m^2}{2}} \prod_{i=1}^{n+1} \left[\cosh\left(\alpha_i M^2\right) - \frac{m^2}{2M^2} \sinh\left(\alpha_i M^2\right)\right].$$

Integration over $\xi_1,\ldots,\xi_{n+1}$ yields a sum of integrals over λ which are once more given by modified Bessel functions.

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